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Shape invariance and laddering equations for the associated hypergeometric functions

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Abstract
Introducing the associated hypergeometric functions in terms of two non-negative integers, we factorize their corresponding differential equation into a product of first-order differential operators by four different ways as shape invariance equations. These shape invariances are realized by four different types of raising and lowering operators. This procedure gives four different pairs of recursion relations on the associated hypergeometric functions.

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1. Introduction

Hypergeometric series and functions play an important role in a wide variety of problems in physics, applied mathematics, engineering and statistics [1–7]. The significance of the hypergeometric differential equations is the fact that every ordinary second-order differential equation with at most three regular singular points can be converted to a hypergeometric differential equation. Another aspect of the mentioned fact is that most of the associated special functions are transformed to the hypergeometric functions by choosing special values for their parameters, appropriate change of variable or suitable change of function. Hypergeometric polynomials are involved in the classical eigenfunctions of singular Sturm–Liouville equations [8–13]. The wavefunctions of quantum mechanics and the correlation functions of some integrable systems are described in terms of the hypergeometric functions [12, 14–21]. Moreover, the hypergeometric functions are used in constructing mathematical models for a large number of physical and chemical phenomena [10, 12, 22–24]. On the other hand, the factorization method has been known as a powerful technique for solving second-order differential equations—in connection with the physical models with orthogonal bases—by means of the raising and lowering operators. The factorization method was first introduced by Darboux [25] and then it was used by Schrödinger [26, 27] for obtaining the solutions of the differential equations. In [27], Schrödinger has suggested four kinds of special factorizations for factorizing the ordinary hypergeometric differential equation. Infeld and Hull [28] extended the method for obtaining the analytic solutions of a certain class of non-relativistic second-order Hamiltonians by means of the creation and annihilation operators. Meanwhile as a new research, a general linearization formula has been found for a product of the hypergeometric polynomials [29]. In [30], Laha *et al* have introduced two special first-order differential operators for generating recursion relations on the hypergeometric functions. These operators also generate the equations derived from the action of the supersymmetry generators on the positive energy solution of the Coulomb field in addition to factorizing the associated second-order differential operator. Their results on the confluent hypergeometric functions lead to derivation of the ladder operators corresponding to the Coulomb–Green wavefunction with boundary condition.

On the basis of the factorization of the associated hypergeometric functions differential equation with respect to two indices, i.e. (a) n, degree of the hypergeometric polynomials (b) m, dependence index for the associated hypergeometric functions, Cotfas [16] has mentioned new ladder operators which increase or decrease the indices n and m by one unit simultaneously. The explicit differential forms of the operators have not been deduced. In fact in this paper, we complete the discussions presented in [16]. In this paper we introduce the associated hypergeometric functions in terms of indices n and m so that their corresponding differential equation is factorized into a product of first-order differential operators in four different ways as shape invariance equations for the indices (n, m) and $(n-1, m)$, (n, m) and $(n, m-1)$, $(n, m-1)$ and $(n-1, m)$ as well as (n, m) and $(n-1, m-1)$. The shape invariance relations (in which the values of n and m do not change) are realized by the ladder operators shifting only n, shifting only m, shifting indices n and m simultaneously and inversely and shifting indices n and m simultaneously and agreeably (in which both indices are lowered or both indices are raised), respectively. Meanwhile for every shape invariance, a pair of recursion relations on the associated hypergeometric functions is derived.

2. Shape invariance equations with respect to *n* and *m*

Let us first consider a second-order linear differential operator for given real parameters $\alpha, \beta > -1$ and $\omega > 0$ as

$$\mathcal{L}^{(\alpha,\beta)}(x) := x^{-\alpha}(1-\omega x)^{-\beta}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\alpha+1}(1-\omega x)^{\beta+1}\frac{\mathrm{d}}{\mathrm{d}x}\right). \tag{1}$$

Lemma 1. *$\mathcal{L}^{(\alpha,\beta)}(x)$ has the following properties:*

(a) It is a Hermitian operator with respect to an inner product with the weight function $W(x) = x^{\alpha}(1-\omega x)^{\beta}$ in the interval $x \in \left(0, \frac{1}{\omega}\right)$.

(b) The action of the operator $\mathcal{L}^{(\alpha,\beta)}(x)$ on an arbitrary polynomial is such that the degree of the polynomial is not increased.

(c) If we show the eigenfunctions of the operator $\mathcal{L}^{(\alpha,\beta)}(x)$ with $F_n^{(\alpha,\beta)}(x)$ as a polynomial exactly of degree n, then we can conclude its eigenvalue equation as follows:

$$\begin{aligned} &x(1-\omega x)F_n''^{(\alpha,\beta)}(x) + [\alpha+1-(\alpha+\beta+2)\omega x]F_n'^{(\alpha,\beta)}(x) \\ &\qquad + n\omega(\alpha+\beta+n+1)F_n^{(\alpha,\beta)}(x) = 0. \end{aligned} \tag{2}$$

Proof. The proof is straightforward. □

Equation (2) is known as the differential equation of hypergeometric polynomials, and it is also obtained from the general form of hypergeometric differential equation

$$y(1-y)\,{}_2\ddot{F}_1(a,b;c;y)+[c-(a+b+1)y]\,{}_2\dot{F}_1(a,b;c;y)-ab\,{}_2F_1(a,b;c;y)=0 \qquad 0<y<1 \tag{3}$$

by choosing the following special values:

$$a=-n \qquad b=\alpha+\beta+n+1 \qquad c=\alpha+1 \qquad y=\omega x. \tag{4}$$

The symbols of prime and dot denote the differentiation with respect to x and y, respectively. Differential equation (3) is the prototype of the Fuchsian equation with three regular singularities, namely 0, 1 and ∞. The hypergeometric function ${}_2F_1(a,b;c;y)$ has the following known expansion:

$${}_2F_1(a,b;c;y)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{y^k}{k!} \qquad (a)_k=a(a+1)\cdots(a+k-1) \tag{5}$$

which is derived by using the series expansion procedure.

Lemma 2. *The hypergeometric polynomials as particular solutions of (2) have a representation of the so-called Rodrigues formula:*

$$F_n^{(\alpha,\beta)}(x)=\frac{a_n(\alpha,\beta)}{x^\alpha(1-\omega x)^\beta}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n(x^{\alpha+n}(1-\omega x)^{\beta+n}) \tag{6}$$

where $a_n(\alpha,\beta)$ are the normalization coefficients.

Proof. See [12]. □

It is easily seen that the coefficient of the highest power of x, x^n, for $F_n^{(\alpha,\beta)}(x)$ is

$$F_n^{(\alpha,\beta)}(x)=a_n(\alpha,\beta)(-\omega)^n\frac{\Gamma(\alpha+\beta+2n+1)}{\Gamma(\alpha+\beta+n+1)}x^n+O(x^{n-1}). \tag{7}$$

Comparing with (5), for the parameters given as (4), we find that

$$F_n^{(\alpha,\beta)}(x)=\frac{\Gamma(\alpha+n+1)a_n(\alpha,\beta)}{\Gamma(\alpha+1)\Gamma(n+1)}\,{}_2F_1(-n,\alpha+\beta+n+1;\alpha+1;\omega x). \tag{8}$$

Lemma 3. *We have*

$$\int_0^{\frac{1}{\omega}}F_n^{(\alpha,\beta)}(x)F_{n'}^{(\alpha,\beta)}(x)x^\alpha(1-\omega x)^\beta\,\mathrm{d}x=\delta_{nn'}h_n^2(\alpha,\beta) \tag{9}$$

where

$$h_n^2(\alpha,\beta)=\frac{a_n^2(\alpha,\beta)}{\omega^{\alpha+1}}\frac{\Gamma(n+1)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}. \tag{10}$$

Proof. This follows immediately from integration by parts. □

Now, we are going to introduce the associated hypergeometric functions with a Rodrigues representation labelled by two non-negative integers.

Lemma 4. *We have the following associated hypergeometric differential equation:*

$$x(1-\omega x)F''^{(\alpha,\beta)}_{n,m}(x) + [\alpha+1-(\alpha+\beta+2)\omega x]F'^{(\alpha,\beta)}_{n,m}(x) + \left[n\omega(\alpha+\beta+n+1) + \frac{m[2(\alpha-\beta)\omega x-(2\alpha+m)]}{4x(1-\omega x)}\right]F^{(\alpha,\beta)}_{n,m}(x) = 0 \qquad 0 \leqslant m \leqslant n \tag{11}$$

with the solutions as

$$F^{(\alpha,\beta)}_{n,m}(x) = \frac{a_{n,m}(\alpha,\beta)}{x^{\alpha+\frac{m}{2}}(1-\omega x)^{\beta+\frac{m}{2}}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m}(x^{\alpha+n}(1-\omega x)^{\beta+n}) \tag{12}$$

where the real constant $a_{n,m}(\alpha,\beta)$ is the normalization coefficient of the associated hypergeometric function $F^{(\alpha,\beta)}_{n,m}(x)$.

Proof. By differentiating the hypergeometric polynomials differential equation (2) m times we get a new differential equation similar to (2), but with new parameters $\alpha+m$, $\beta+m$ and $n-m$ instead of α, β and n, respectively. Thus for the obtained differential equation, we have a polynomial solution of degree $n-m$ as $F^{(\alpha+m,\beta+m)}_{n-m}(x)$. Then it is trivial to show that the associated hypergeometric functions

$$F^{(\alpha,\beta)}_{n,m}(x) = \frac{a_{n,m}(\alpha,\beta)}{a_{n-m}(\alpha+m,\beta+m)}x^{\frac{m}{2}}(1-\omega x)^{\frac{m}{2}}F^{(\alpha+m,\beta+m)}_{n-m}(x) \tag{13}$$

satisfy the differential equation (11). □

Obviously by choosing $m=0$, the associated hypergeometric functions differential equation (11) converts to the differential equation corresponding to the hypergeometric polynomials given in relation (2).

Lemma 5. *We have*

$$\int_0^{\frac{1}{\omega}} F^{(\alpha,\beta)}_{n,m}(x)F^{(\alpha,\beta)}_{n',m}(x)x^{\alpha}(1-\omega x)^{\beta}\,\mathrm{d}x = \delta_{nn'}h^2_{n,m}(\alpha,\beta) \qquad n,n' \geqslant m \tag{14}$$

where

$$h^2_{n,m}(\alpha,\beta) = \frac{a^2_{n,m}(\alpha,\beta)}{\omega^{\alpha+m+1}}\frac{\Gamma(n-m+1)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+m+1)}. \tag{15}$$

Proof. The proof follows by using lemma 3 and formula (13). □

Lemma 5 implies that the associated hypergeometric functions with different n, for a given m, form an orthogonal set with respect to an inner product with the weight function $x^{\alpha}(1-\omega x)^{\beta}$ in the interval $x \in \left(0,\frac{1}{\omega}\right)$. Evidently, equation (14) also expresses that the norms of the associated hypergeometric functions, $h_{n,m}(\alpha,\beta)$, are determined by specifying the normalization coefficients $a_{n,m}(\alpha,\beta)$. Satisfaction of the raising and lowering relations obtained from the shape invariance with respect to the parameters n and m will determine the normalization coefficients $a_{n,m}(\alpha,\beta)$.

According to the discussions in [14, 15, 31], the shape invariance with respect to one parameter leads to not only a representation of the supersymmetry algebra but also a representation of the parasupersymmetry algebra of arbitrary order. It is worth emphasizing that the simultaneous shape invariance with respect to more than one parameter also leads to a representation of the (para)supersymmetry algebra. Preferably, the mentioned discussions should be followed in the context of the quantum mechanical problems. Thus, in what follows, we explain shape invariance with respect to two parameters n and m, separately, only from the mathematical point of view.

Theorem 1. *The associated hypergeometric functions differential equation (11) is factorized into a product of first-order differential operators as (a) shape invariance equations (of the first type) with respect to n, i.e. as equations* (n, m) *and* $(n-1, m)$,

$$\begin{aligned} A_+(n, m; x)A_-(n, m; x)F_{n,m}^{(\alpha,\beta)}(x) &= E(n, m)F_{n,m}^{(\alpha,\beta)}(x) \\ A_-(n, m; x)A_+(n, m; x)F_{n-1,m}^{(\alpha,\beta)}(x) &= E(n, m)F_{n-1,m}^{(\alpha,\beta)}(x) \end{aligned} \tag{16}$$

with

$$\begin{aligned} A_+(n, m; x) &= x(1-\omega x)\frac{\mathrm{d}}{\mathrm{d}x} - (\alpha+\beta+n)\omega x + \frac{1}{2}(2\alpha+n) - \frac{(n-m)(\alpha-\beta)}{2(\alpha+\beta+2n)} \\ A_-(n, m; x) &= -x(1-\omega x)\frac{\mathrm{d}}{\mathrm{d}x} - n\omega x + \frac{n}{2} - \frac{(n-m)(\alpha-\beta)}{2(\alpha+\beta+2n)} \end{aligned} \tag{17}$$

$$E(n, m) = \frac{(n-m)(\alpha+n)(\beta+n)(\alpha+\beta+n+m)}{(\alpha+\beta+2n)^2} \tag{18}$$

(b) shape invariance equations (of second type) with respect to m, i.e. as equations (n, m) *and* $(n, m-1)$,

$$\begin{aligned} A_+(m; x)A_-(m; x)F_{n,m}^{(\alpha,\beta)}(x) &= \mathcal{E}(n, m)F_{n,m}^{(\alpha,\beta)}(x) \\ A_-(m; x)A_+(m; x)F_{n,m-1}^{(\alpha,\beta)}(x) &= \mathcal{E}(n, m)F_{n,m-1}^{(\alpha,\beta)}(x) \end{aligned} \tag{19}$$

with

$$\begin{aligned} A_+(m; x) &= \sqrt{x(1-\omega x)}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{(m-1)(2\omega x-1)}{2\sqrt{x(1-\omega x)}} \\ A_-(m; x) &= -\sqrt{x(1-\omega x)}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{2(\alpha+\beta+m)\omega x - 2\alpha - m}{2\sqrt{x(1-\omega x)}} \end{aligned} \tag{20}$$

$$\mathcal{E}(n, m) = (n-m+1)(\alpha+\beta+n+m)\omega. \tag{21}$$

Proof. The proof of the factorizations (16) and (19) can be derived by means of a direct substitution of the explicit forms of $A_\pm(n, m; x)$, $E(n, m)$, $A_\pm(m; x)$ and $\mathcal{E}(n, m)$. In other words, one can easily verify that each of the relations given in (16) and (19) is a copy of the associated hypergeometric differential equation (11). The technical proofs of the first and second types of the factorizations can be found in [31] and [15, 16], respectively. □

It is also seen that the operators $A_+(m; x)$ and $A_-(m; x)$ ($A_+(n, m; x)$ and $A_-(n, m; x)$) are (not) the Hermitian conjugates of each other with respect to the inner product (14).

3. Simultaneous realization of laddering equations with respect to *n* and *m*

Now regarding the shape invariance equations (16) and (19) we can obtain the raising and lowering relations of the indices n and m of the associated hypergeometric functions $F_{n,m}^{(\alpha,\beta)}(x)$. It is clear that realization of equations (16) and (19) does not impose any condition on the normalization coefficients $a_{n,m}(\alpha, \beta)$. However, realization of the laddering equations with respect to n and m imposes two recursion relations with respect to n and m, respectively, on the coefficients.

Theorem 2. *For a given m, the raising and lowering relations of the index n,*

$$A_+(n,m;x)F^{(\alpha,\beta)}_{n-1,m}(x) = \sqrt{E(n,m)}F^{(\alpha,\beta)}_{n,m}(x) \tag{22a}$$

$$A_-(n,m;x)F^{(\alpha,\beta)}_{n,m}(x) = \sqrt{E(n,m)}F^{(\alpha,\beta)}_{n-1,m}(x) \tag{22b}$$

and for a given n, the raising and lowering relations of the index m,

$$A_+(m;x)F^{(\alpha,\beta)}_{n,m-1}(x) = \sqrt{\mathcal{E}(n,m)}F^{(\alpha,\beta)}_{n,m}(x) \tag{23a}$$

$$A_-(m;x)F^{(\alpha,\beta)}_{n,m}(x) = \sqrt{\mathcal{E}(n,m)}F^{(\alpha,\beta)}_{n,m-1}(x) \tag{23b}$$

are simultaneously established if the normalization coefficient $a_{n,m}(\alpha,\beta)$ is chosen as

$$a_{n,m}(\alpha,\beta) = (-1)^m\omega^{\frac{m}{2}}\sqrt{\frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(n-m+1)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}}C(\alpha,\beta)$$
$$0 \leqslant m \leqslant n < \infty \tag{24}$$

in which $C(\alpha,\beta)$ is an arbitrary real constant independent of n and m.

Proof. Applying equation (13) in equation (22*a*) and dividing both sides by the factor $x^{\frac{m}{2}}(1-\omega x)^{\frac{m}{2}}$ then, by means of comparing the coefficients of the highest power of x, x^{n-m}, one may obtain

$$a_{n,m}(\alpha,\beta) = \sqrt{\frac{\alpha+\beta+n+m}{(n-m)(\alpha+n)(\beta+n)}}a_{n-1,m}(\alpha,\beta) \qquad n > m. \tag{25}$$

If we follow a similar procedure in connection with equation (22*b*) and compare the coefficients of the highest power of x, x^{n-m+1}, on both sides then we will obtain a relation which is just an identity. Repeated application of the recursion relation (25) for a given m leads to

$$a_{n,m}(\alpha,\beta) = \sqrt{\frac{\Gamma(\alpha+\beta+n+m+1)\Gamma(\alpha+m+1)\Gamma(\beta+m+1)}{\Gamma(n-m+1)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+2m+1)}}a_{m,m}(\alpha,\beta)$$
$$n \geqslant m. \tag{26}$$

Also, using equation (13) in equation (23*a*) and dividing both sides by the factor $x^{\frac{m}{2}}(1-\omega x)^{\frac{m}{2}}$ then, by comparing the coefficients of the highest power of x, x^{n-m}, the following recursion relation is obtained:

$$a_{n,m}(\alpha,\beta) = -\frac{a_{n,m+1}(\alpha,\beta)}{\sqrt{(n-m)(\alpha+\beta+n+m+1)\omega}} \qquad m \leqslant n-1. \tag{27}$$

If we follow the same procedure for equation (23*b*) then we have to divide both sides by $x^{\frac{m-1}{2}}(1-\omega x)^{\frac{m-1}{2}}$ and compare the coefficients of x^{n-m+1} on both sides. Finally, we will get the recursion relation (27) again. Relation (27), for a given n, immediately gives

$$a_{n,m}(\alpha,\beta) = \left(-\frac{1}{\sqrt{\omega}}\right)^{n-m}\sqrt{\frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(n-m+1)\Gamma(\alpha+\beta+2n+1)}}a_{n,n}(\alpha,\beta) \qquad m \leqslant n. \tag{28}$$

In fact, equations (26) and (28) are two different constraints on the normalization coefficients of the associated hypergeometric functions which have two free indices. Clearly, by comparing them, it appears that

$$a_{n,n}(\alpha,\beta) = (-1)^n\omega^{\frac{n}{2}}\sqrt{\frac{\Gamma(\alpha+\beta+2n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}}C(\alpha,\beta) \qquad n = 0,1,2,\ldots. \tag{29}$$

Note that equation (29) is also valid when $n = m$. Thus, using equations (26) and (28) we can get the same expression (24) for the normalization coefficients. □

For a given m, the laddering relations (22) are infinite since $m \leqslant n < \infty$, while for a given n, the laddering relations (23) are finite since $0 \leqslant m \leqslant n$. In deriving relation (24) we have not used any data involved in equation (22*b*). However, one can verify relation (22*b*) by applying (24) to it. Similar discussions can be carried out for the associated Jacobi functions[3]. Now, we follow the discussion from another point of view. In other words, after determining the normalization coefficients as relation (24), we investigate the norm of the associated hypergeometric functions.

Corollary 1. *For a given n and m, the norm of the associated hypergeometric functions* $F_{n,m}^{(\alpha,\beta)}(x)$ *is independent of m, and its square is*

$$h_{n,m}^2(\alpha,\beta) = \frac{C^2(\alpha,\beta)}{\omega^{\alpha+1}(\alpha+\beta+2n+1)}. \tag{30}$$

Proof. It follows immediately by substituting equation (24) into (15). □

Each of the laddering equations (22) and (23) separately proposes an algebraic solution for deriving the associated hypergeometric functions.

[3] If we define a second-order linear differential operator, for given real parameters $\alpha, \beta > -1$ and $\omega > 0$, in the interval $z \in (\frac{-1}{\omega}, \frac{1}{\omega})$ as

$$\mathcal{L}^{(\alpha,\beta)}(z) := (1-\omega z)^{-\alpha}(1+\omega z)^{-\beta}\frac{\mathrm{d}}{\mathrm{d}z}\left((1-\omega z)^{\alpha+1}(1+\omega z)^{\beta+1}\frac{\mathrm{d}}{\mathrm{d}z}\right)$$

then we can obtain the differential equation corresponding to the associated Jacobi functions $P_{n,m}^{(\alpha,\beta)}(z)$ (of hypergeometric type) as follows:

$$(1-\omega^2 z^2)P''^{(\alpha,\beta)}_{n,m}(z) - \omega[\alpha-\beta+\omega(\alpha+\beta+2)z]P'^{(\alpha,\beta)}_{n,m}(z) + \omega^2\left[n(\alpha+\beta+n+1) - \frac{m(\alpha+\beta+m+(\alpha-\beta)\omega z)}{1-\omega^2 z^2}\right]P_{n,m}^{(\alpha,\beta)}(z) = 0.$$

Note that in this footnote, the prime symbol indicates differentiation with respect to z. The associated Jacobi functions $P_{n,m}^{(\alpha,\beta)}(z)$ as the solutions of the above differential equation have the following Rodrigues representation:

$$P_{n,m}^{(\alpha,\beta)}(z) = \frac{b_{n,m}(\alpha,\beta)}{(1-\omega z)^{\alpha+\frac{m}{2}}(1+\omega z)^{\beta+\frac{m}{2}}}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n-m}((1-\omega z)^{\alpha+n}(1+\omega z)^{\beta+n})$$

in which $b_{n,m}(\alpha,\beta)$ are the normalization coefficients. Clearly by using the change of variable $\omega x = \frac{1-\omega z}{2}$, the interval $z \in (-\frac{1}{\omega}, \frac{1}{\omega})$ converts to the interval $x \in (0, \frac{1}{\omega})$, and the associated Jacobi differential equation reduces to (11). This leads, in turn, to

$$P_{n,m}^{(\alpha,\beta)}(z) \propto F_{n,m}^{(\alpha,\beta)}(x).$$

Comparing equation (12) with the Rodrigues formula for $P_{n,m}^{(\alpha,\beta)}(z)$, the above proportionality converts to the following equation:

$$P_{n,m}^{(\alpha,\beta)}(z) = (-1)^{n-m}2^n\omega^{n-\frac{m}{2}}\frac{b_{n,m}(\alpha,\beta)}{a_{n,m}(\alpha,\beta)}F_{n,m}^{(\alpha,\beta)}(x).$$

If we perform the factorization with respect to n and m separately and simultaneously, then similar to the associated hypergeometric functions we get the following result for the associated Jacobi functions:

$$b_{n,m}(\alpha,\beta) = \left(\frac{-1}{2}\right)^n\left(\frac{-1}{\omega}\right)^{n-m}\sqrt{\frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(n-m+1)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}}C(\alpha,\beta) \qquad 0 \leqslant m \leqslant n < \infty.$$

Thus, using the formalism introduced in this paper, one can follow the discussions for the associated Jacobi functions instead of the associated hypergeometric functions.

Corollary 2. *We have the following algebraic solutions for the associated hypergeometric differential equation (11):*

$$F_{n,m}^{(\alpha,\beta)}(x) = \frac{A_+(n,m;x)A_+(n-1,m;x)\cdots A_+(m+1,m;x)F_{m,m}^{(\alpha,\beta)}(x)}{\sqrt{E(n,m)E(n-1,m)\cdots E(m+1,m)}} \qquad n \geqslant m+1 \tag{31}$$

$$F_{n,m}^{(\alpha,\beta)}(x) = \frac{A_-(m+1;x)A_-(m+2;x)\cdots A_-(n;x)F_{n,n}^{(\alpha,\beta)}(x)}{\sqrt{\mathcal{E}(n,m+1)\mathcal{E}(n,m+2)\cdots\mathcal{E}(n,n)}} \qquad m \leqslant n-1 \tag{32}$$

where

$$F_{m,m}^{(\alpha,\beta)}(x) = a_{m,m}(\alpha,\beta)x^{\frac{m}{2}}(1-\omega x)^{\frac{m}{2}}. \tag{33}$$

Proof. To present the proof of the corollary, it is sufficient to consider equations $E(m,m) = \mathcal{E}(n,n+1) = 0$ and then obtain the following first-order differential equations from (22*b*) and (23*a*):

$$A_-(m,m;x)F_{m,m}^{(\alpha,\beta)}(x) = 0 \tag{34}$$

$$A_+(n+1;x)F_{n,n}^{(\alpha,\beta)}(x) = 0. \tag{35}$$

The solution of the first-order differential equation (34) is (33), and the solution of (35) is (33) if m is replaced by n. For a given m and n, by repeated application of the raising and lowering relations (22*a*) and (23*b*), one may get the arbitrary associated hypergeometric function $F_{n,m}^{(\alpha,\beta)}(x)$ as equations (31) and (32), respectively. □

Note that the algebraic solution (33) is in agreement with the analytic solution (12). The laddering equations (22) and (23) corresponding to the parameters n and m, respectively, show that there are two pairs of recursion relations of first and second types on three associated hypergeometric functions.

Corollary 3. *There exist the following two independent recursion relations (of first type) on the index n for the associated hypergeometric functions*

$$\begin{aligned}
&\left[-(\alpha+\beta+2n+1)\omega x + \frac{2\alpha+2n+1}{2} - \frac{(n-m)(\alpha-\beta)}{2(\alpha+\beta+2n)} - \frac{(n-m+1)(\alpha-\beta)}{2(\alpha+\beta+2n+2)}\right]F_{n,m}^{(\alpha,\beta)}(x)\\
&\qquad = \sqrt{E(n,m)}F_{n-1,m}^{(\alpha,\beta)}(x) + \sqrt{E(n+1,m)}F_{n+1,m}^{(\alpha,\beta)}(x)\\
&\left[2x(1-\omega x)\frac{\mathrm{d}}{\mathrm{d}x} - (\alpha+\beta+1)\omega x + \frac{2\alpha+1}{2} + \frac{(n-m)(\alpha-\beta)}{2(\alpha+\beta+2n)}\right.\\
&\qquad\left. - \frac{(n-m+1)(\alpha-\beta)}{2(\alpha+\beta+2n+2)}\right]F_{n,m}^{(\alpha,\beta)}(x)\\
&\qquad = \sqrt{E(n+1,m)}F_{n+1,m}^{(\alpha,\beta)}(x) - \sqrt{E(n,m)}F_{n-1,m}^{(\alpha,\beta)}(x).
\end{aligned} \tag{36}$$

Proof. In order to derive these recursion relations it is sufficient to change n to $n+1$ in equation (22*a*); then the obtained result must be added to and subtracted from (22*b*). □

Corollary 4. *There are the following two independent recursion relations (of second type) on the index m for the associated hypergeometric functions*

$$\begin{aligned}
&\frac{(\alpha+\beta+2m)\omega x - \alpha - m}{\sqrt{x(1-\omega x)}}F_{n,m}^{(\alpha,\beta)}(x) = \sqrt{\mathcal{E}(n,m)}F_{n,m-1}^{(\alpha,\beta)}(x) + \sqrt{\mathcal{E}(n,m+1)}F_{n,m+1}^{(\alpha,\beta)}(x)\\
&\left[2\sqrt{x(1-\omega x)}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{(\alpha+\beta)\omega x - \alpha}{\sqrt{x(1-\omega x)}}\right]F_{n,m}^{(\alpha,\beta)}(x)\\
&\qquad = \sqrt{\mathcal{E}(n,m+1)}F_{n,m+1}^{(\alpha,\beta)}(x) - \sqrt{\mathcal{E}(n,m)}F_{n,m-1}^{(\alpha,\beta)}(x).
\end{aligned} \tag{37}$$

Proof. The proof is quite similar to the proof of corollary 3. □

4. Shape invariance and laddering equations with respect to *n* and *m* simultaneously

The laddering equations (22) and (23), which shift n and m separately, lead to the derivation of two types of new factorizations for the associated hypergeometric differential equation (11) so that, on that basis, shape invariance equations may be written for the indices $(n, m-1)$ and $(n-1, m)$ as well as (n, m) and $(n-1, m-1)$. Each of these shape invariances is realized by a pair of the first-order differential operators whose corresponding laddering equations shift both of the indices n and m, simultaneously and inversely as well as simultaneously and agreeably, respectively.

Theorem 3. *Let us define two new differential operators as*

$$\begin{aligned}A_{+,-}(n, m; x) &:= A_{-}(m; x)A_{+}(n, m; x) - A_{+}(n, m-1; x)A_{-}(m; x)\\ A_{-,+}(n, m; x) &:= A_{-}(n, m; x)A_{+}(m; x) - A_{+}(m; x)A_{-}(n, m-1; x).\end{aligned} \tag{38}$$

(a) They satisfy the raising and lowering relations with respect to n and m, simultaneously as

$$A_{+,-}(n, m; x)F^{(\alpha,\beta)}_{n-1,m}(x) = \sqrt{\frac{\omega(n-m+1)E(n, m)}{\alpha+\beta+n+m}}\,F^{(\alpha,\beta)}_{n,m-1}(x) \tag{39a}$$

$$A_{-,+}(n, m; x)F^{(\alpha,\beta)}_{n,m-1}(x) = \sqrt{\frac{\omega(n-m+1)E(n, m)}{\alpha+\beta+n+m}}\,F^{(\alpha,\beta)}_{n-1,m}(x). \tag{39b}$$

So, the operator $A_{+,-}(n, m; x)$ increases n and decreases m however, the operator $A_{-,+}(n, m; x)$ decreases n and increases m.

(b) They satisfy shape invariance equations (of third type) with respect to the indices n and m as equations $(n, m-1)$ and $(n-1, m)$:

$$\begin{aligned}A_{+,-}(n, m; x)A_{-,+}(n, m; x)F^{(\alpha,\beta)}_{n,m-1}(x) &= \frac{\omega(n-m+1)E(n, m)}{\alpha+\beta+n+m}F^{(\alpha,\beta)}_{n,m-1}(x)\\ A_{-,+}(n, m; x)A_{+,-}(n, m; x)F^{(\alpha,\beta)}_{n-1,m}(x) &= \frac{\omega(n-m+1)E(n, m)}{\alpha+\beta+n+m}F^{(\alpha,\beta)}_{n-1,m}(x).\end{aligned} \tag{40}$$

(c) They have the following explicit forms as the first-order differential operators

$$\begin{aligned}A_{+,-}(n, m; x) &= \frac{1}{2}\left[\frac{\beta-\alpha}{\alpha+\beta+2n} - (1-2\omega x)\right]\sqrt{x(1-\omega x)}\frac{\mathrm{d}}{\mathrm{d}x} + (n-m)\omega\sqrt{x(1-\omega x)}\\ &\quad + \frac{1}{4}\left[\frac{\beta-\alpha}{\alpha+\beta+2n} - (1-2\omega x)\right]\frac{-2(\alpha+\beta+m)\omega x + 2\alpha + m}{\sqrt{x(1-\omega x)}}\\ A_{-,+}(n, m; x) &= -\frac{1}{2}\left[\frac{\beta-\alpha}{\alpha+\beta+2n} - (1-2\omega x)\right]\\ &\quad \times\sqrt{x(1-\omega x)}\frac{\mathrm{d}}{\mathrm{d}x} + (n-m+1)\omega\sqrt{x(1-\omega x)}\\ &\quad + \frac{1}{4}\left[\frac{\beta-\alpha}{\alpha+\beta+2n} - (1-2\omega x)\right]\frac{(m-1)(1-2\omega x)}{\sqrt{x(1-\omega x)}}.\end{aligned} \tag{41}$$

Proof. In order to prove the laddering relations (39), it is sufficient to use the laddering relations (22) and (23) in the definitions (38). The proof of the shape invariance equations (40) is trivial by direct substitution of the laddering relations (39) into them. The explicit differential forms of the operators $A_{\pm,\mp}(n, m; x)$ are also calculated by equations (17) and (20). □

Corollary 5. *There are two independent recursion relations (of third type) among three associated hypergeometric functions as*

$$\begin{aligned}
&\left[\frac{(\alpha-\beta)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}\sqrt{x(1-\omega x)}\frac{\mathrm{d}}{\mathrm{d}x}+(2n-2m+1)\omega\sqrt{x(1-\omega x)}\right.\\
&\qquad+\frac{1}{4}\left[\frac{\beta-\alpha}{\alpha+\beta+2n+2}-(1-2\omega x)\right]\frac{-2(\alpha+\beta+m)\omega x+m+2\alpha}{\sqrt{x(1-\omega x)}}\\
&\qquad\left.+\frac{1}{4}\left[\frac{\beta-\alpha}{\alpha+\beta+2n}-(1-2\omega x)\right]\frac{m(1-2\omega x)}{\sqrt{x(1-\omega x)}}\right]F_{n,m}^{(\alpha,\beta)}(x)\\
&\quad=\sqrt{\frac{\omega(n-m+2)E(n+1,m)}{\alpha+\beta+n+m+1}}F_{n+1,m-1}^{(\alpha,\beta)}(x)\\
&\qquad+\sqrt{\frac{\omega(n-m)E(n,m+1)}{\alpha+\beta+n+m+1}}F_{n-1,m+1}^{(\alpha,\beta)}(x)\\
&\left[\left(\frac{(\beta-\alpha)(\alpha+\beta+2n+1)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}-(1-2\omega x)\right)\sqrt{x(1-\omega x)}\frac{\mathrm{d}}{\mathrm{d}x}+\omega\sqrt{x(1-\omega x)}\right.\\
&\qquad+\frac{1}{4}\left[\frac{\beta-\alpha}{\alpha+\beta+2n+2}-(1-2\omega x)\right]\frac{-2(\alpha+\beta+m)\omega x+2\alpha+m}{\sqrt{x(1-\omega x)}}\\
&\qquad\left.-\frac{1}{4}\left[\frac{\beta-\alpha}{\alpha+\beta+2n}-(1-2\omega x)\right]\frac{m(1-2\omega x)}{\sqrt{x(1-\omega x)}}\right]F_{n,m}^{(\alpha,\beta)}(x)\\
&\quad=\sqrt{\frac{\omega(n-m+2)E(n+1,m)}{\alpha+\beta+n+m+1}}F_{n+1,m-1}^{(\alpha,\beta)}(x)\\
&\qquad-\sqrt{\frac{\omega(n-m)E(n,m+1)}{\alpha+\beta+n+m+1}}F_{n-1,m+1}^{(\alpha,\beta)}(x).
\end{aligned}\tag{42}$$

Proof. To prove this corollary one may change n and m to $n+1$ and $m+1$ in equations (39*a*) and (39*b*), respectively, and then the obtained results should be added to and subtracted from each other. □

Theorem 4. *Let us define two new differential operators as*

$$\begin{aligned}
A_{+,+}(n,m;x) &:= A_{+}(m;x)A_{+}(n,m-1;x)-A_{+}(n,m;x)A_{+}(m;x)\\
A_{-,-}(n,m;x) &:= A_{-}(n,m-1;x)A_{-}(m;x)-A_{-}(m;x)A_{-}(n,m;x).
\end{aligned}\tag{43}$$

(a) They satisfy the raising and lowering relations with respect to n and m, simultaneously as

$$A_{+,+}(n,m;x)F_{n-1,m-1}^{(\alpha,\beta)}(x)=\sqrt{\frac{\omega(\alpha+\beta+n+m-1)E(n,m)}{n-m}}F_{n,m}^{(\alpha,\beta)}(x)\tag{44a}$$

$$A_{-,-}(n,m;x)F_{n,m}^{(\alpha,\beta)}(x)=\sqrt{\frac{\omega(\alpha+\beta+n+m-1)E(n,m)}{n-m}}F_{n-1,m-1}^{(\alpha,\beta)}(x).\tag{44b}$$

Hence, the operator $A_{+,+}(n,m;x)$ increases both of the indices n and m, however, the operator $A_{-,-}(n,m;x)$ decreases both of them.

(b) They satisfy shape invariance equations (of fourth type) with respect to the indices n and m as equations (n,m) and $(n-1,m-1)$:

$$\begin{aligned}A_{+,+}(n,m;x)A_{-,-}(n,m;x)F_{n,m}^{(\alpha,\beta)}(x) &= \frac{\omega(\alpha+\beta+n+m-1)E(n,m)}{n-m}F_{n,m}^{(\alpha,\beta)}(x)\\ A_{-,-}(n,m;x)A_{+,+}(n,m;x)F_{n-1,m-1}^{(\alpha,\beta)}(x) &= \frac{\omega(\alpha+\beta+n+m-1)E(n,m)}{n-m}F_{n-1,m-1}^{(\alpha,\beta)}(x).\end{aligned} \tag{45}$$

(c) They have the following explicit forms as the first-order differential operators:

$$\begin{aligned}A_{+,+}(n,m;x) &= \frac{1}{2}\left[\frac{\beta-\alpha}{\alpha+\beta+2n}+(1-2\omega x)\right]\sqrt{x(1-\omega x)}\,\frac{\mathrm{d}}{\mathrm{d}x}\\ &\quad-(\alpha+\beta+n+m-1)\omega\sqrt{x(1-\omega x)}\\ &\quad-\frac{1}{4}\left[\frac{\beta-\alpha}{\alpha+\beta+2n}+(1-2\omega x)\right]\frac{(m-1)(1-2\omega x)}{\sqrt{x(1-\omega x)}}\\ A_{-,-}(n,m;x) &= -\frac{1}{2}\left[\frac{\beta-\alpha}{\alpha+\beta+2n}+(1-2\omega x)\right]\sqrt{x(1-\omega x)}\,\frac{\mathrm{d}}{\mathrm{d}x}\\ &\quad-(\alpha+\beta+n+m)\omega\sqrt{x(1-\omega x)}\\ &\quad-\frac{1}{4}\left[\frac{\beta-\alpha}{\alpha+\beta+2n}+(1-2\omega x)\right]\frac{-2(\alpha+\beta+m)\omega x+2\alpha+m}{\sqrt{x(1-\omega x)}}.\end{aligned} \tag{46}$$

Proof. In the same way as presented in the proof of theorem 3, one can deduce the relations (44), (45) and (46). □

It is noted that equations (44) and (45) have no singular point when $n = m$. Similar to the shape invariance equations (16) and (19), one may check that each of the equations given in (40) and (45) can be converted to the differential equation (11) for the associated hypergeometric functions by some manipulations. In fact, the relations (16), (19), (40) and (45) as shape invariance equations are different types of the factorizations for (11). In figure 1, we have schematically shown all the associated hypergeometric functions $F_{n,m}^{(\alpha,\beta)}(x)$ as points (n, m) with $0 \leqslant m \leqslant n < \infty$ in the flat plane with n and m as the horizontal and vertical axes, respectively. The ladder operators $A_{\pm}(m; x)$, $A_{\pm}(n, m; x)$, $A_{\pm,\pm}(n, m; x)$ and $A_{\pm,\mp}(n, m; x)$ displace the associated hypergeometric functions lain on the vertical and horizontal lines, as well as the lines parallel to the bisectors of the first and fourth quadrants, respectively.

Corollary 6. *There exist two independent recursion relations (of fourth type) among three associated hypergeometric functions as*

$$\begin{aligned}&\left[\frac{(\alpha-\beta)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}\sqrt{x(1-\omega x)}\frac{\mathrm{d}}{\mathrm{d}x}-(2\alpha+2\beta+2n+2m+1)\omega\sqrt{x(1-\omega x)}\right.\\ &\qquad-\frac{1}{4}\left[\frac{\beta-\alpha}{\alpha+\beta+2n}+(1-2\omega x)\right]\frac{-2(\alpha+\beta+m)\omega x+2\alpha+m}{\sqrt{x(1-\omega x)}}\\ &\qquad\left.-\frac{1}{4}\left[\frac{\beta-\alpha}{\alpha+\beta+2n+2}+(1-2\omega x)\right]\frac{m(1-2\omega x)}{\sqrt{x(1-\omega x)}}\right]F_{n,m}^{(\alpha,\beta)}(x)\\ &\quad=\sqrt{\frac{\omega(\alpha+\beta+n+m+1)E(n+1,m+1)}{n-m}}F_{n+1,m+1}^{(\alpha,\beta)}(x)\\ &\qquad+\sqrt{\frac{\omega(\alpha+\beta+n+m-1)E(n,m)}{n-m}}F_{n-1,m-1}^{(\alpha,\beta)}(x)\end{aligned}$$

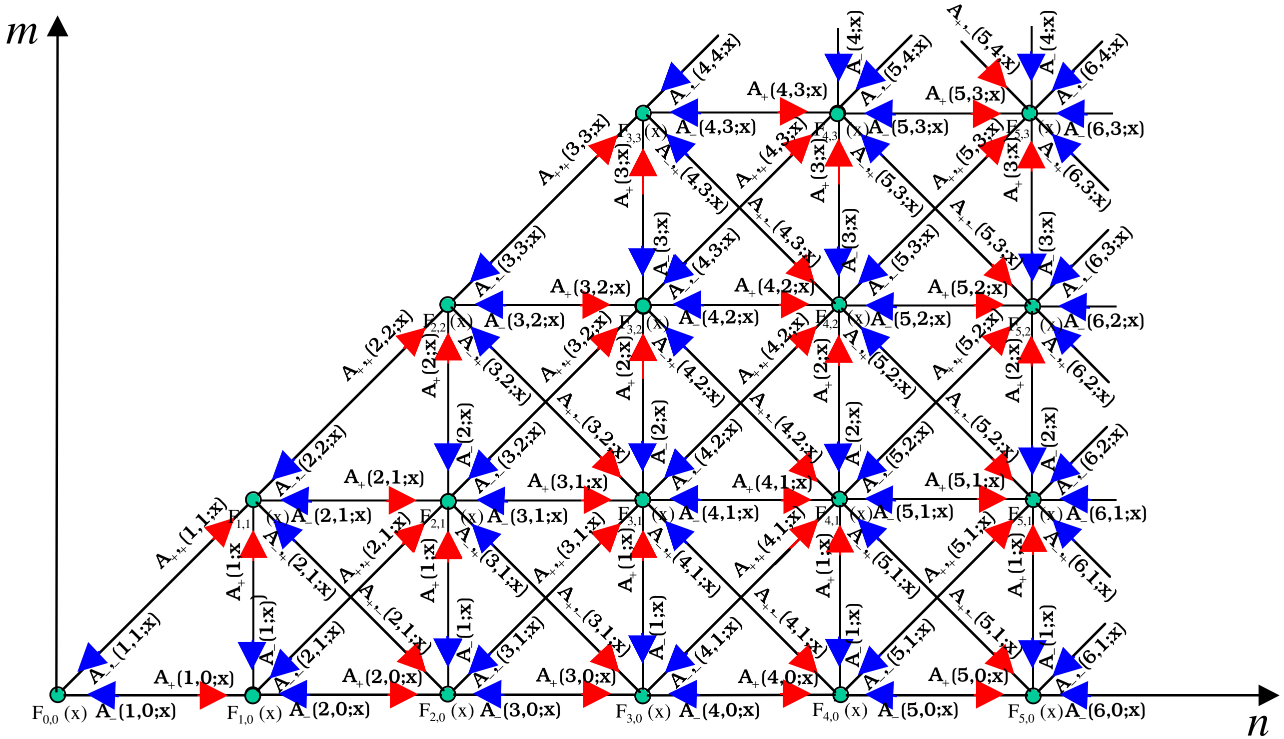

Figure 1. The plane of displacements of the associated hypergeometric functions in four different ways by the ladder operators shifting only *n*, shifting only *m*, shifting indices *n* and *m* simultaneously and inversely and shifting indices *n* and *m* simultaneously and agreeably.

$$\left[\left(\frac{(\beta-\alpha)(\alpha+\beta+2n+1)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}+(1-2\omega x)\right)\sqrt{x(1-\omega x)}\frac{\mathrm{d}}{\mathrm{d}x}-\omega\sqrt{x(1-\omega x)}\right.$$
$$+\frac{1}{4}\left[\frac{\beta-\alpha}{\alpha+\beta+2n}+(1-2\omega x)\right]\frac{-2(\alpha+\beta+m)\omega x+2\alpha+m}{\sqrt{x(1-\omega x)}}$$
$$\left.-\frac{1}{4}\left[\frac{\beta-\alpha}{\alpha+\beta+2n+2}+(1-2\omega x)\right]\frac{m(1-2\omega x)}{\sqrt{x(1-\omega x)}}\right]F_{n,m}^{(\alpha,\beta)}(x)$$
$$=\sqrt{\frac{\omega(\alpha+\beta+n+m+1)E(n+1,m+1)}{n-m}}F_{n+1,m+1}^{(\alpha,\beta)}(x)$$
$$-\sqrt{\frac{\omega(\alpha+\beta+n+m-1)E(n,m)}{n-m}}F_{n-1,m-1}^{(\alpha,\beta)}(x). \tag{47}$$

Proof. In order to prove this corollary it is sufficient to increase each of the indices n and m by one unit in equation (44*a*), then the obtained result must be added to and subtracted from equation (44*b*). □

Note that two pairs of recursion relations obtained in the relations (42) and (47) are different from (36) and (37). In fact, both of the recursion relations (42) and (47) involve the derivative of the associated hypergeometric functions, however, the first recursion relations of (36) and (37) do not have the terms involving the derivative of associated hypergeometric functions. Obviously, the results of theorems 3 and 4 take simple forms when $\alpha = \beta$. It is also clear that the discussions of this paper can be followed for the other associated special functions, for example the associated Jacobi functions, which are transformed to the associated hypergeometric functions by well-known methods. However, quantum states of some of the 1D solvable models such as the trigonometric Pöschl–Teller, Natanzon and trigonometric Scarf are calculated by using the hypergeometric-type differential equation in terms of the associated hypergeometric functions. One can apply realization of the simultaneous laddering relations with respect to more than one parameter by the quantum states and show that they represent supersymmetry algebra with higher $\mathcal{N}$. This is a problem that one may study.

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